IB Groups, Rings, and Modules // Example Sheet 2

All rings in this course are commutative and have a multiplicative identity.

1. Let $\omega = \frac{1}{2}(1+\sqrt{-3}) \in \mathbb{C}$, let $R = \{a + b\omega : \ a, b \in \mathbb{Z}\}$, and let $F = \{a + b\omega : \ a, b \in \mathbb{Q}\}$. Show that R is a subring of $\mathbb{C}$, and that F is a subfield of $\mathbb{C}$. What are the units of R?

2. An element r of a ring R is called *nilpotent* if $r^n = 0$ for some n.
   (i) What are the nilpotent elements of $\mathbb{Z}/6\mathbb{Z}$? Of $\mathbb{Z}/8\mathbb{Z}$? Of $\mathbb{Z}/24\mathbb{Z}$? Of $\mathbb{Z}/180\mathbb{Z}$?
   (ii) Show that if r is nilpotent then r is not a unit, but $1+r$ and $1-r$ are units.
   (iii) Show that set of the nilpotent elements form an ideal N of R. What are the nilpotent elements in the quotient ring R/N?

3. Let r be an element of a ring R. Show that the polynomial $1 + rX \in R[X]$ is a unit if and only if r is nilpotent. Is it possible for the polynomial $1 + X$ to be a product of two non-units?

4. Let $I_1 \subset I_2 \subset I_3 \subset \cdots$ be ideals in a ring R. Show that the union $I = \bigcup_{n=1}^{\infty} I_n$ is also an ideal. If each I_n is proper, explain why I must be proper.

5. Show that if I and J are ideals in the ring R, then so is $I \cap J$, and the quotient $R/(I \cap J)$ is isomorphic to a subring of the product $R/I \times R/J$. Show further that if there exist $x \in I$ and $y \in J$ with $x + y = 1$ then $R/(I \cap J) \cong R/I \times R/J$. What does this result say when $R = \mathbb{Z}$?

6. Let R be an integral domain. Show that a polynomial in $R[X]$ of degree d can have at most d roots. Deduce that the natural ring homomorphism from $R[X]$ to the ring of all functions $R \to R$ is injective if and only if R is infinite. Give an example of a monic quadratic polynomial in $(\mathbb{Z}/8\mathbb{Z})[X]$ that has more than two roots.

7. Write down a prime ideal in $\mathbb{Z} \times \mathbb{Z}$ that is not maximal. Explain why in a finite ring all prime ideals are maximal.

8. Explain why, for p a prime number, there is a unique ring of order p. How many rings are there of order 4?

9. Let R be an integral domain and F be its field of fractions. Suppose that $\phi : R \to K$ is an injective ring homomorphism from R to a field K. Show that ϕ extends to an injective homomorphism $\Phi : F \to K$ from F to K. What happens if we do not assume that ϕ is injective?

10. An element r of a ring R is called *idempotent* if $r^2 = r$.
    (i) What are the idempotent elements of $\mathbb{Z}/6\mathbb{Z}$? Of $\mathbb{Z}/8\mathbb{Z}$? Of $\mathbb{Z}/24\mathbb{Z}$? How many idempotents are there in $\mathbb{Z}/180\mathbb{Z}$?
    (ii) Show that if r is idempotent then so is $r' = 1 - r$, and that $rr' = 0$. Show also that the ideal (r) is naturally a ring, and that R is isomorphic as a ring to $(r) \times (r')$.

11. Let F be a field, and let $R = F[X, Y]$ be the polynomial ring in two variables.
    (i) Let I be the principal ideal $(X - Y)$ of R. Show that $R/I \cong F[X]$.
    (ii) Describe R/I when $I = (X^2 + Y)$.
    (iii) What can you say about $R/(X^2 - Y^2)$? Is it an integral domain? Does it have nilpotent or idempotent elements?

Optional Questions

12. Suppose a ring R has the property that for each $x \in R$ there is a $n \geq 2$ such that $x^n = x$. Show that every prime ideal of R is maximal.

13. This question illustrates a construction of the real numbers, so you should avoid mentioning them in your answer.

    A sequence $\{a_n\}$ of rational numbers is a *Cauchy sequence* if $|a_n - a_m| \to 0$ as $m, n \to \infty$, and $\{a_n\}$ is a *null sequence* if $a_n \to 0$ as $n \to \infty$. Quoting any standard results from Analysis, show that the set of Cauchy sequences with componentwise addition and multiplication form a ring C, and that the null sequences form a maximal ideal N.

    Deduce that C/N is a field, which contains a subfield which may be identified with $\mathbb{Q}$. Explain briefly why the equation $x^2 = 2$ has a solution in this field.

14. Let R be the set of all functions from $\mathbb{R}^2$ to $\mathbb{R}$. It can be given the structure of a ring by pointwise addition and multiplication of functions.

    (a) Is the ring R an integral domain?

    (b) Let $C \subset R$ be the subset of continuous functions. Check that C forms a subring. Is it an integral domain?

    Let $I \subset R$ be the subset of all functions whose value at $(0, 0)$ is equal to 0. Prove that I is a maximal ideal.

15. Let ϖ be a set of prime numbers. Write $\mathbb{Z}_\varpi$ for the collection of all rationals m/n (in lowest terms) such that the only prime factors of the denominator n are in ϖ.

    (i) Show that $\mathbb{Z}_\varpi$ is a subring of the field $\mathbb{Q}$ of rational numbers.

    (ii) Show that any subring R of $\mathbb{Q}$ is of the form $\mathbb{Z}_\varpi$ for some set ϖ of primes.

    (iii) Given (ii), what are the maximal subrings of $\mathbb{Q}$?

16. (a) Show that the set $\mathbb{P}(S)$ of all subsets of a given set S is a ring with respect to the operations of symmetric difference and intersection. Describe the principal ideals in this ring. Show that the ideal (A, B) generated by elements A, B is in fact principal.

    (b) Let $\mathbb{P}'(S)$ be the set of functions $S \to \mathbb{Z}/2\mathbb{Z}$. Check that this set of functions forms a ring under pointwise addition and multiplication of functions. Show that $\mathbb{P}'(S)$ and $\mathbb{P}(S)$ are isomorphic.

    (c) A ring R is called *Boolean* if every element of R is idempotent. Prove that every finite Boolean ring is isomorphic to a power-set ring $\mathbb{P}(S)$ for some set S. Give an example to show that this need not remain true for infinite Boolean rings.

Comments or corrections to `dr508@cam.ac.uk`